

# Gauged twistor spinors and symmetry operators

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We consider gauged twistor spinors which are supersymmetry generators of supersymmetric and superconformal field theories in curved backgrounds. We show that the spinor bilinears of gauged twistor spinors satisfy the gauged conformal Killing-Yano equation. We prove that the symmetry operators of the gauged twistor spinor equation can be constructed from ordinary conformal Killing-Yano forms in constant curvature backgrounds. This provides a way to obtain gauged twistor spinors from ordinary twistor spinors.

## I. INTRODUCTION

Supersymmetric field theories in flat backgrounds can be generalized to curved backgrounds by coupling them with supergravity and taking the gravity multiplets non-dynamic [1]. Preserved supersymmetries in those cases are found by calculating the vanishing condition of the gravitino variation under supersymmetry transformations. Similarly, superconformal field theories can also be extended to curved backgrounds by coupling to conformal supergravity and they are also studied using holography [2–5]. In all of these cases, supersymmetry generators are determined by the same equation which comes from the variation of the gravitino. It is called the gauged twistor equation or charged conformal Killing spinor equation. It is a generalization of the twistor equation written in terms of the gauged covariant derivative and the gauged Dirac operator [6–8]. Because of the existence of the gauge fields, the solutions of the spinor field equations correspond to  $\text{Spin}^c$  spinors, namely spinor fields are sections of the bundle that is written as a product of spinor bundle and gauge bundle. The solutions of the gauged twistor equation are called gauged twistor spinors and generate the preserved supersymmetries of supersymmetric and superconformal field theories in curved backgrounds.

One of the methods for finding the solutions of an equation is constructing the symmetry operators of it. Symmetry operators take a solution of the equation and give another solution. The set of mutually commuting symmetry operators are used for finding a general solution by using the method of separation variables [9]. Symmetry operators of some basic spinor field equations can be constructed from the hidden symmetries of the background manifold. Hidden symmetries are defined as the antisymmetric generalizations of Killing vector fields and conformal Killing vector fields to higher degree differential forms. For Killing vector fields, those generalizations are called Killing-Yano (KY) forms and for conformal Killing vector fields, they are conformal Killing-Yano (CKY) forms. The symmetry operators of massless and massive Dirac equations are constructed out of CKY forms and KY forms, respectively [10–15]. Similarly, symmetry operators of geometric Killing spinors are written in terms of odd degree KY forms in constant curvature backgrounds [16]. CKY forms are used in the construction of the symmetry operators of twistor spinors in constant curvature backgrounds and normal CKY forms play the same role in Einstein manifolds [17]. Moreover, the symmetry operators of Killing and twistor spinors are also used in the definitions of more general structures such as extended Killing superalgebras and extended conformal superalgebras [16, 17].

In this paper, we consider the gauged twistor equation and find its integrability conditions in general  $n$  dimensions. We write the spinor bilinears of gauged twistor spinors and show that they correspond to gauged CKY forms which are generalizations of CKY forms with respect to a gauged covariant derivative. We propose a symmetry operator for the gauged twistor equation in terms of CKY forms and prove that it satisfies the required conditions of being a symmetry operator in constant curvature manifolds. Since the spinor bilinears of ordinary twistor spinors correspond to CKY forms, this also opens a way to find the solutions of the gauged twistor equation by using ordinary twistor spinors. This provides a way to find the supersymmetry generators of supersymmetric and superconformal field theories in constant curvature backgrounds.

The paper is organized as follows. We define the gauged twistor equation and find its integrability conditions in Section 2. In Section 3, we construct the spinor bilinears of gauged twistor spinors and show that they correspond to gauged CKY forms. A symmetry operator of gauged twistor spinors is proposed in Section 4 and it is proved that

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it satisfies the symmetry operator requirements in constant curvature backgrounds. We also show that the symmetry operator can be written in terms of ordinary twistor spinors. Section 5 concludes the paper.

## II. GAUGED TWISTOR SPINORS

On a manifold  $M$  with  $\text{Spin}^c$ -structure, one can define a bundle of  $U(1)$ -valued spinors  $S \otimes \Sigma$  where  $S$  is the spinor bundle and  $\Sigma$  is the  $U(1)$  bundle. These types of manifolds can be used to model the backgrounds of supersymmetric field theories on curved space-time. Supersymmetric field theories in flat space-time can be extended to curved space-times by coupling with conformal supergravity and then fixing the gravity multiplets. To preserve some amount of supersymmetry in curved backgrounds, one obtains a condition on the supersymmetry parameters that comes from the variation of the gravitino. The supersymmetry parameters must satisfy the following gauged twistor (or charged conformal Killing spinor) equation in  $n$  dimensions;

$$\widehat{\nabla}_X \psi = \frac{1}{n} \tilde{X} \cdot \widehat{\mathcal{D}} \psi \quad (1)$$

with respect to any vector field  $X$  and its metric dual  $\tilde{X}$  where  $\psi$  is a  $\text{Spin}^c$  (or  $U(1)$ -valued) spinor. Gauged spinor covariant derivative  $\widehat{\nabla}_X$  with respect to  $X$  is defined in terms of the spinor covariant derivative  $\nabla_X$  and gauge connection 1-form  $A$ , which is generally complex, as

$$\widehat{\nabla}_X := \nabla_X + i_X A \quad (2)$$

where  $i_X$  is the interior derivative or contraction operation with respect to  $X$ . The Dirac operator  $\mathcal{D}$  is defined from spinor covariant derivative, frame basis  $X_a$  and co-frame basis  $e^a$  with the property  $e^a(X_b) = \delta_b^a$  as  $\mathcal{D} = e^a \cdot \nabla_{X_a}$  where  $\cdot$  denotes the Clifford product. So, the gauged Dirac operator  $\widehat{\mathcal{D}}$  in (1) is written as follows

$$\widehat{\mathcal{D}} := e^a \cdot \widehat{\nabla}_{X_a} = \mathcal{D} + A \quad (3)$$

where we have used the expansion of Clifford product in terms of wedge product and interior derivative as  $x \cdot \alpha = x \wedge \alpha + i_{\tilde{x}} \alpha$  for any 1-form  $x$ , its metric dual  $\tilde{x}$  and any differential  $p$ -form  $\alpha$ . We have also used the property  $e^a \wedge i_{X_a} \alpha = p \alpha$  [18].

The exterior derivative  $d$  and co-derivative  $\delta$  can be written in terms of covariant derivative (with vanishing torsion) as

$$d = e^a \wedge \nabla_{X_a} \quad , \quad \delta = -i_{X^a} \nabla_{X_a} \quad (4)$$

and gauged exterior derivative  $\widehat{d}$  and co-derivative  $\widehat{\delta}$  can be written in terms of them

$$\widehat{d} := e^a \wedge \widehat{\nabla}_{X_a} = d + A \wedge \quad (5)$$

$$\widehat{\delta} := -i_{X^a} \widehat{\nabla}_{X_a} = \delta - i_{\tilde{A}} \quad (6)$$

where  $\tilde{A}$  is the vector field that is metric dual of the 1-form  $A$ . However, on the contrary to the case of  $d$  and  $\delta$  which satisfy  $d^2 = \delta^2 = 0$ , the squares of gauged exterior and co-derivatives are written in the following form

$$\widehat{d}^2 = F \wedge \quad (7)$$

$$\widehat{\delta}^2 = -(i_{X^a} i_{X^b} F) i_{X_a} i_{X_b} \quad (8)$$

where  $F = dA$  is the curvature of the gauge connection 1-form  $A$  [19].

### A. Integrability conditions

The existence of gauged twistor spinors in a manifold depends on some integrability conditions of (1) which constrain the curvature characteristics of the background manifold. They can be obtained by taking second covariant derivatives of the gauged twistor equation and by using the following definition of the curvature operator of the gauged covariant derivative

$$\widehat{R}(X, Y) = [\widehat{\nabla}_X, \widehat{\nabla}_Y] - \widehat{\nabla}_{[X, Y]} \quad (9)$$

where  $X$  and  $Y$  are arbitrary vector fields. From the definition (2), it can be written in terms of the curvature operator  $R(X, Y)$  of the Levi-Civita connection and the curvature  $F$  of the gauge connection as follows

$$\widehat{R}(X_a, X_b) = R(X_a, X_b) + i_{X_b} i_{X_a} F \quad (10)$$

where  $\{X_a\}$  is an orthonormal frame. The action of the curvature operator  $R(X_a, X_b)$  on a spinor  $\psi$  can be written in terms of curvature 2-forms  $R_{ab}$  as  $R(X_a, X_b)\psi = \frac{1}{2}R_{ab}.\psi$  [18, 19]. From the curvature 2-forms  $R_{ab}$ , the definition of Ricci 1-forms  $P_a$  and curvature scalar  $\mathcal{R}$  can be stated as  $P_a = i_{X_b} R_{ba}$  and  $\mathcal{R} = i_{X_a} P_a$ , respectively.

The action of the operator in (10) on a gauged twistor spinor  $\psi$  must be equal to the action of the right hand side of (9) on the same gauged twistor spinor. By using (1), we obtain

$$R(X_a, X_b)\psi + (i_{X_b} i_{X_a} F)\psi = \frac{1}{n} \widehat{\nabla}_{X_a} (e_b \cdot \widehat{\mathcal{D}}\psi) - \frac{1}{n} \widehat{\nabla}_{X_b} (e_a \cdot \widehat{\mathcal{D}}\psi). \quad (11)$$

So, one can write the action of curvature 2-forms  $R_{ab}$  on gauged twistor spinors as follows

$$R_{ab}.\psi = \frac{2}{n} \left( e_b \cdot \widehat{\nabla}_{X_a} \widehat{\mathcal{D}}\psi - e_a \cdot \widehat{\nabla}_{X_b} \widehat{\mathcal{D}}\psi \right) - 2(i_{X_b} i_{X_a} F)\psi. \quad (12)$$

For zero torsion, we have the equalities  $R_{ab} \wedge e^a = 0$  and  $e^a \cdot R_{ab} = P_b$ . By using them, the action of Ricci 1-forms  $P_a$  on gauged twistor spinors can be calculated from (12)

$$\begin{aligned} P_b.\psi &= \frac{2}{n} \left( e^a \cdot e_b \cdot \widehat{\nabla}_{X_a} \widehat{\mathcal{D}}\psi - e^a \cdot e_a \cdot \widehat{\nabla}_{X_b} \widehat{\mathcal{D}}\psi \right) - 2e^a \cdot (i_{X_b} i_{X_a} F)\psi \\ &= -\frac{2}{n} e_b \cdot \widehat{\mathcal{D}}^2 \psi - \frac{2(n-2)}{n} \widehat{\nabla}_{X_b} \widehat{\mathcal{D}}\psi + 2(i_{X_b} F).\psi \end{aligned} \quad (13)$$

where we have used the Clifford algebra identity  $e^a \cdot e_b + e_b \cdot e^a = 2g_b^a$  for the metric  $g_{ab}$  and the definition (3). Similarly, Ricci 1-forms satisfy the equalities  $P_a \wedge e^a = 0$  and  $e^a \cdot P_a = \mathcal{R}$ . So, we can write the action of the curvature scalar  $\mathcal{R}$  on gauged twistor spinors from (13)

$$\begin{aligned} \mathcal{R}\psi &= -\frac{2}{n} e^a \cdot e_a \cdot \widehat{\mathcal{D}}^2 \psi - \frac{2(n-2)}{n} e^a \cdot \widehat{\nabla}_{X_a} \widehat{\mathcal{D}}\psi + 2e^a \cdot i_{X_a} F.\psi \\ &= -\frac{4(n-1)}{n} \widehat{\mathcal{D}}^2 \psi + 4F.\psi. \end{aligned} \quad (14)$$

By combining (13) and (14), one can obtain the following two integrability conditions of the gauged twistor equation

$$\widehat{\mathcal{D}}^2 \psi = -\frac{n}{4(n-1)} \mathcal{R}\psi + \frac{n}{n-1} F.\psi \quad (15)$$

$$\widehat{\nabla}_{X_a} \widehat{\mathcal{D}}\psi = \frac{n}{2} K_a.\psi - \frac{n}{(n-1)(n-2)} e_a \cdot F.\psi + \frac{n}{n-2} i_{X_a} F.\psi \quad (16)$$

where the 1-form  $K_a$  is defined as follows

$$K_a = \frac{1}{n-2} \left( \frac{\mathcal{R}}{2(n-1)} e_a - P_a \right). \quad (17)$$

Moreover, from the definition of the conformal 2-forms

$$C_{ab} = R_{ab} - \frac{1}{n-2} (P_a \wedge e_b - P_b \wedge e_a) + \frac{1}{(n-1)(n-2)} \mathcal{R} e_{ab} \quad (18)$$

where  $e_{ab} = e_a \wedge e_b$ , the third integrability condition that corresponds to the action of  $C_{ab}$  on gauged twistor spinors can be found from (12), (13) and (14) as

$$C_{ab}.\psi = 2(i_{X_a} i_{X_b} F)\psi + \frac{2}{n-2} (e_b \cdot i_{X_a} F - e_a \cdot i_{X_b} F).\psi + \frac{4}{(n-1)(n-2)} e_a \cdot e_b \cdot F.\psi. \quad (19)$$

For  $A = 0$ , (15), (16) and (19) reduce to the integrability conditions of the ordinary twistor spinor equation [20–22].

### III. SPINOR BILINEARS AND GAUGED CKY FORMS

The tensor product of the spinor space  $S$  and dual spinor space  $S^*$  correspond to the algebra of endomorphisms over the spinor space  $S \otimes S^* = \text{End}S$  and it is isomorphic to the Clifford algebra of the relevant dimension which is also isomorphic to the exterior algebra  $\Lambda M$  of differential forms on  $M$ . So, the tensor products of spinors and its duals which are called spinor bilinears can be written as a sum of different degree differential forms

$$\psi \otimes \bar{\psi} = (\psi, \psi) + (\psi, e_a \cdot \psi) e^a + (\psi, e_{ba} \cdot \psi) e^{ab} + \dots + (\psi, e_{a_p \dots a_2 a_1} \cdot \psi) e^{a_1 a_2 \dots a_p} + \dots + (-1)^{\lfloor n/2 \rfloor} (\psi, z \cdot \psi) z \quad (20)$$

where  $e^{a_1 a_2 \dots a_p} = e^{a_1} \wedge e^{a_2} \wedge \dots \wedge e^{a_p}$ ,  $\lfloor \cdot \rfloor$  is the floor function that takes the integer part of the argument,  $z$  is the volume form and  $(\cdot, \cdot)$  denotes the spinor inner product. Every  $p$ -form component on the right hand side of (20) is called the  $p$ -form Dirac current as the generalization of the Dirac current that corresponds to the metric dual of the 1-form part of the spinor bilinear [23].  $p$ -form Dirac currents will be denoted as follows

$$(\psi \bar{\psi})_p = (\psi, e_{a_p \dots a_2 a_1} \cdot \psi) e^{a_1 a_2 \dots a_p}. \quad (21)$$

For a gauged twistor spinor  $\psi$ , by requiring that the connection  $\hat{\nabla}$  is compatible with the spinor inner product  $(\cdot, \cdot)$ , we will show that the  $p$ -form Dirac currents of gauged twistor spinors satisfy the gauged CKY equation which is the generalization of the CKY equation that corresponds to the antisymmetric generalization of the conformal Killing equation to higher degree forms.

After applying the gauged covariant derivative to (21) and doing some manipulations, one obtains that

$$\begin{aligned} \hat{\nabla}_{X_a} (\psi \bar{\psi})_p &= \left( (\hat{\nabla}_{X_a} \psi) \bar{\psi} \right)_p + \left( \psi \overline{\hat{\nabla}_{X_a} \psi} \right)_p \\ &= \frac{1}{n} \left( (e_a \cdot \hat{\mathcal{D}} \psi) \bar{\psi} \right)_p + \frac{1}{n} \left( \psi \overline{e_a \cdot \hat{\mathcal{D}} \psi} \right)_p \\ &= \frac{1}{n} \left( e_a \cdot \hat{\mathcal{D}} (\psi \bar{\psi}) \right)_p - \frac{1}{n} \left( e_a \cdot e_b \cdot \psi \overline{\hat{\nabla}_{X_b} \psi} \right)_p + \frac{1}{n} \left( \psi \overline{\hat{\nabla}_{X_b} \psi} \cdot e_b \cdot e_a \right)_p \end{aligned}$$

where we have used (1) and  $(\hat{\nabla}_{X_a} \psi) \bar{\psi} = \hat{\nabla}_{X_a} (\psi \bar{\psi}) - \psi \overline{(\hat{\nabla}_{X_a} \psi)}$  with the definition  $\hat{\mathcal{D}} = e_a \cdot \hat{\nabla}_{X_a}$  on differential forms. From (5) and (6), one can write  $\hat{\mathcal{D}} = \hat{d} - \hat{\delta}$  and obtain the following equality by using the definition of the Clifford product of a 1-form with any  $p$ -form in terms of the wedge product and interior derivative

$$\begin{aligned} \hat{\nabla}_{X_a} (\psi \bar{\psi})_p &= \frac{1}{n} \left( e_a \wedge \hat{d} (\psi \bar{\psi})_{p-2} + i_{X_a} \hat{d} (\psi \bar{\psi})_p \right) - \frac{1}{n} \left( e_a \wedge (e_b \cdot \psi \overline{\hat{\nabla}_{X^b} \psi})_{p-1} + i_{X_a} (e_b \cdot \psi \overline{\hat{\nabla}_{X^b} \psi})_{p+1} \right) \\ &\quad - \frac{1}{n} \left( e_a \wedge \hat{\delta} (\psi \bar{\psi})_p + i_{X_a} \hat{\delta} (\psi \bar{\psi})_{p+2} \right) \pm \frac{1}{n} \left( e_a \wedge (\psi \overline{\hat{\nabla}_{X^b} \psi} \cdot e_b)_{p-1} + i_{X_a} (\psi \overline{\hat{\nabla}_{X^b} \psi} \cdot e_b)_{p+1} \right) \end{aligned} \quad (22)$$

where  $\pm$  sign depends on the chosen inner automorphism of the Clifford algebra which is used in the definition of the duality operation  $\cdot$ . By wedge multiplying (22) with  $e^a$  from the left and using (5), we can write

$$\hat{d} (\psi \bar{\psi})_p = \frac{p+1}{n} \left( \hat{d} (\psi \bar{\psi})_p - \hat{\delta} (\psi \bar{\psi})_{p+2} \right) - \frac{p+1}{n} \left( (e_b \cdot \psi \overline{\hat{\nabla}_{X^b} \psi})_{p+1} \mp (\psi \overline{\hat{\nabla}_{X^b} \psi} \cdot e_b)_{p+1} \right) \quad (23)$$

and similarly by taking the interior derivative of (22) with respect to  $X_a$  and using (6), it can also be written

$$\hat{\delta} (\psi \bar{\psi})_p = -\frac{n-p+1}{n} \left( \hat{d} (\psi \bar{\psi})_{p-2} - \hat{\delta} (\psi \bar{\psi})_p \right) + \frac{n-p+1}{n} \left( (e_b \cdot \psi \overline{\hat{\nabla}_{X^b} \psi})_{p-1} \mp (\psi \overline{\hat{\nabla}_{X^b} \psi} \cdot e_b)_{p-1} \right). \quad (24)$$

So, by comparing (22), (23) and (24), one can see that the  $p$ -form Dirac currents of gauged twistor spinors satisfy the following equation

$$\hat{\nabla}_{X_a} (\psi \bar{\psi})_p = \frac{1}{p+1} i_{X_a} \hat{d} (\psi \bar{\psi})_p - \frac{1}{n-p+1} e_a \wedge \hat{\delta} (\psi \bar{\psi})_p. \quad (25)$$

This equation is called the gauged CKY equation. In general, a  $p$ -form  $\omega$  is called a gauged CKY  $p$ -form, if it satisfies the following gauged CKY equation

$$\hat{\nabla}_{X_a} \omega = \frac{1}{p+1} i_{X_a} \hat{d} \omega - \frac{1}{n-p+1} e_a \wedge \hat{\delta} \omega. \quad (26)$$

From (2), it can also be written in terms of the Levi-Civita connection  $\nabla$  and the gauge potential 1-form  $A$  as

$$\begin{aligned} & \nabla_{X_a}\omega - \frac{1}{p+1}i_{X_a}d\omega + \frac{1}{n-p+1}e_a \wedge \delta\omega \\ &= -\frac{p}{p+1}(i_{X_a}A)\omega - \frac{1}{p+1}A \wedge i_{X_a}\omega + \frac{1}{n-p+1}e_a \wedge i_{\tilde{A}}\omega. \end{aligned} \quad (27)$$

For  $A = 0$ , it reduces to the ordinary CKY equation which is the antisymmetric generalization of the conformal Killing equation to higher degree forms

$$\nabla_{X_a}\omega = \frac{1}{p+1}i_{X_a}d\omega - \frac{1}{n-p+1}e_a \wedge \delta\omega. \quad (28)$$

For  $p = 1$ , (27) reduces to the shear-free vector field equation which is the generalization of the conformal Killing equation and describes the vector fields that constitute shear-free congruences [19].

Integrability conditions of the gauged CKY equation can be calculated by taking second covariant derivatives of (26). After some manipulations, they can be obtained as follows

$$\begin{aligned} \widehat{\nabla}_{X_b}\widehat{d}\omega &= \frac{p+1}{p}R_{ab} \wedge i_{X^a}\omega + \frac{p+1}{p(n-p+1)}e_b \wedge \widehat{d}\widehat{d}\omega \\ &\quad + i_{X_b}F \wedge \omega - \frac{1}{p}F \wedge i_{X_b}\omega + \frac{p+1}{p(n-p+1)}e_b \wedge A \wedge \widehat{d}\omega \end{aligned} \quad (29)$$

$$\begin{aligned} \widehat{\nabla}_{X_b}\widehat{\delta}\omega &= \frac{n-p+1}{n-p} \left( (i_{X_a}P_b)i_{X^a}\omega + i_{X_a}R_{cb} \wedge i_{X^c}i_{X^a}\omega + (i_{X_b}i_{X_a}F)i_{X^a}\omega \right) \\ &\quad - \frac{n-p+1}{(p+1)(n-p)}i_{X_b}\widehat{\delta}\widehat{d}\omega - (i_{X_b}A)\widehat{\delta}\omega - \frac{1}{n-p}e_b \wedge \left( i_{\tilde{A}}\widehat{\delta}\omega + (i_{X_a}i_{X^c}F)i_{X^a}i_{X^c}\omega \right) \end{aligned} \quad (30)$$

and their combination gives

$$\begin{aligned} \frac{p}{p+1}\widehat{\delta}\widehat{d}\omega + \frac{n-p}{n-p+1}\widehat{d}\widehat{\delta}\omega &= P_a \wedge i_{X^a}\omega + R_{ab} \wedge i_{X^a}i_{X^b}\omega \\ &\quad - \frac{n-p}{n-p+1}A \wedge \widehat{\delta}\omega + i_{X^a}F \wedge i_{X^a}\omega. \end{aligned} \quad (31)$$

For  $A = 0$ , they reduce to the integrability conditions of the ordinary CKY equation which read as [24, 25]

$$\nabla_{X_b}d\omega = \frac{p+1}{p}R_{ab} \wedge i_{X^a}\omega + \frac{p+1}{p(n-p+1)}e_b \wedge d\delta\omega \quad (32)$$

$$\nabla_{X_b}\delta\omega = \frac{n-p+1}{n-p} \left( (i_{X_a}P_b)i_{X^a}\omega + i_{X_a}R_{cb} \wedge i_{X^c}i_{X^a}\omega \right) - \frac{n-p+1}{(p+1)(n-p)}i_{X_b}\delta d\omega \quad (33)$$

$$\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1}d\delta\omega = P_a \wedge i_{X^a}\omega + R_{ab} \wedge i_{X^a}i_{X^b}\omega. \quad (34)$$

#### IV. SYMMETRY OPERATORS

Solutions of the gauged twistor equation (1) gives the supersymmetry parameters of supersymmetric field theories coupled with conformal supergravity. So, finding a solution generating technique for (1) is an important problem. Rather than solving an equation directly, one can also construct symmetry operators of it which give the solutions of an equation from a known solution. For example, symmetry operators of massless and massive Dirac equation can be constructed from CKY and KY forms, respectively [10, 11]. Similarly, symmetry operators of geometric Killing spinors and ordinary twistor spinors can also be written in terms of KY and CKY forms respectively in constant

curvature manifolds [16, 17]. We can search for the symmetry operators of the gauged twistor equation in terms of gauged or ordinary CKY forms. We propose the following operator

$$\begin{aligned} L_\omega &= -(-1)^p \frac{p}{n} \omega \cdot \widehat{\mathcal{D}} + \frac{p}{2(p+1)} d\omega + \frac{p}{2(n-p+1)} \delta\omega \\ &= -(-1)^p \frac{p}{n} \omega \cdot \mathcal{D} + \frac{p}{2(p+1)} d\omega + \frac{p}{2(n-p+1)} \delta\omega - (-1)^p \frac{p}{n} \omega \cdot A \end{aligned} \quad (35)$$

written in terms of ordinary CKY  $p$ -forms  $\omega$ . Note that  $d$  and  $\delta$  are exterior and co-derivatives with respect to the Levi-Civita connection and  $\widehat{\mathcal{D}}$  is the gauged Dirac operator (3). (35) reduces to the symmetry operators of ordinary twistor spinors for  $A = 0$ .

To prove that (35) is a symmetry operator for the gauged twistor equation, we need to show that if  $\psi$  is a gauged twistor spinor, then  $L_\omega \psi$  is a solution of the gauged twistor equation, namely it satisfies the following equality

$$\widehat{\nabla}_{X_a} L_\omega \psi = \frac{1}{n} e_a \cdot \widehat{\mathcal{D}} L_\omega \psi \quad (36)$$

which can also be written in terms the Levi-Civita connection as

$$\nabla_{X_a} L_\omega \psi - \frac{1}{n} e_a \cdot \mathcal{D} L_\omega \psi = -\frac{n-2}{2n} e_a \cdot A \cdot L_\omega \psi - \frac{1}{2} A \cdot e_a \cdot L_\omega \psi. \quad (37)$$

So, we will expand all the terms in (37) to check the equality. By using (35), the first term on the left hand side of (37) can be obtained as follows

$$\begin{aligned} \nabla_{X_a} L_\omega \psi &= -(-1)^p \frac{p}{n} \nabla_{X_a} \omega \cdot \widehat{\mathcal{D}} \psi - (-1)^p \frac{p}{n} \omega \cdot \nabla_{X_a} \widehat{\mathcal{D}} \psi + \frac{p}{2(p+1)} \nabla_{X_a} d\omega \cdot \psi \\ &\quad + \frac{p}{2(p+1)} d\omega \cdot \nabla_{X_a} \psi + \frac{p}{2(n-p+1)} \nabla_{X_a} \delta\omega \cdot \psi + \frac{p}{2(n-p+1)} \delta\omega \cdot \nabla_{X_a} \psi. \end{aligned} \quad (38)$$

Here, we can use (1) and (16) which are written in terms of the Levi-Civita connection as

$$\nabla_{X_a} \psi = \frac{1}{n} e_a \cdot \widehat{\mathcal{D}} \psi - \frac{1}{2} (e_a \cdot A + A \cdot e_a) \cdot \psi \quad (39)$$

$$\nabla_{X_a} \widehat{\mathcal{D}} \psi = \frac{n}{2} K_a \cdot \psi - \frac{n}{(n-1)(n-2)} e_a \cdot F \cdot \psi + \frac{n}{n-2} i_{X_a} F \cdot \psi - \frac{1}{2} (e_a \cdot A + A \cdot e_a) \cdot \widehat{\mathcal{D}} \psi. \quad (40)$$

Hence, (38) transforms into the following form

$$\begin{aligned} \nabla_{X_a} L_\omega \psi &= \left[ -(-1)^p \frac{p}{n} \nabla_{X_a} \omega + (-1)^p \frac{p}{2n} (e_a \cdot A + A \cdot e_a) \cdot \omega \right. \\ &\quad \left. + \frac{p}{2n(p+1)} d\omega \cdot e_a + \frac{p}{2n(n-p+1)} \delta\omega \cdot e_a \right] \cdot \widehat{\mathcal{D}} \psi \\ &\quad + \left[ -(-1)^p \frac{p}{2} \omega \cdot K_a + (-1)^p \frac{p}{(n-1)(n-2)} \omega \cdot e_a \cdot F - (-1)^p \frac{p}{n-2} \omega \cdot i_{X_a} F \right. \\ &\quad \left. + \frac{p}{2(p+1)} \nabla_{X_a} d\omega - \frac{p}{4(p+1)} (e_a \cdot A + A \cdot e_a) \cdot d\omega + \frac{p}{2(n-p+1)} \nabla_{X_a} \delta\omega \right. \\ &\quad \left. - \frac{p}{4(n-p+1)} (e_a \cdot A + A \cdot e_a) \cdot \delta\omega \right] \cdot \psi \end{aligned} \quad (41)$$

where we have used the fact that  $e_a \cdot A + A \cdot e_a = 2i_{X_a} A$  is a function and it commutes with the differential forms  $\omega$ ,  $d\omega$  and  $\delta\omega$ . The second term on the left hand side of (37) can also be written from (41) and we obtain

$$\begin{aligned} -\frac{1}{n} e_a \cdot \mathcal{D} L_\omega \psi &= -\frac{1}{n} e_a \cdot e^b \cdot \nabla_{X_b} L_\omega \psi \\ &= \left[ (-1)^p \frac{p}{n^2} e_a \cdot e^b \cdot \nabla_{X_b} \omega - (-1)^p \frac{p}{n^2} e_a \cdot A \cdot \omega \right. \\ &\quad \left. - \frac{p}{2n^2(p+1)} e_a \cdot e^b \cdot d\omega \cdot e_b - \frac{p}{2n^2(n-p+1)} e_a \cdot e^b \cdot \delta\omega \cdot e_b \right] \cdot \widehat{\mathcal{D}} \psi \end{aligned}$$

$$\begin{aligned}
& + \left[ (-1)^p \frac{p}{2n} e_a \cdot e^b \cdot \omega \cdot K_b - (-1)^p \frac{p}{n(n-1)(n-2)} e_a \cdot e^b \cdot \omega \cdot e_b \cdot F \right. \\
& + (-1)^p \frac{p}{n(n-2)} e_a \cdot e^b \cdot \omega \cdot i_{X_b} F - \frac{p}{2n(p+1)} e_a \cdot e^b \cdot \nabla_{X_b} d\omega \\
& \left. + \frac{p}{2n(p+1)} e_a \cdot A \cdot d\omega - \frac{p}{2n(n-p+1)} e_a \cdot e^b \cdot \nabla_{X_b} \delta\omega + \frac{p}{2n(n-p+1)} e_a \cdot A \cdot \delta\omega \right] \cdot \psi \quad (42)
\end{aligned}$$

where we have simplified the terms by using again the relation  $e_a \cdot A + A \cdot e_a = 2i_{X_a} A$  and  $A = (i_{X_a} A)e^a$ . Similarly, by using (35), we can write the terms on the right hand side of (37) in the following way

$$\begin{aligned}
-\frac{n-2}{2n} e_a \cdot A \cdot L_\omega \psi &= (-1)^p \frac{p(n-2)}{2n^2} e_a \cdot A \cdot \omega \cdot \widehat{\mathcal{D}}\psi \\
&- \left[ \frac{p(n-2)}{4n(p+1)} e_a \cdot A \cdot d\omega + \frac{p(n-2)}{4n(n-p+1)} e_a \cdot A \cdot \delta\omega \right] \cdot \psi \quad (43)
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{2} A \cdot e_a \cdot L_\omega \psi &= (-1)^p \frac{p}{2n} A \cdot e_a \cdot \omega \cdot \widehat{\mathcal{D}}\psi \\
&- \left[ \frac{p}{4(p+1)} A \cdot e_a \cdot d\omega + \frac{p}{4(n-p+1)} A \cdot e_a \cdot \delta\omega \right] \cdot \psi. \quad (44)
\end{aligned}$$

Now, we write all the terms in (37) explicitly and we are in a position to check the correctness of (37) by comparing the equalities in (41)-(44). We will do this in two steps, since we can consider the coefficients of  $\widehat{\mathcal{D}}\psi$  and  $\psi$  separately in each equality. So, as a first step, the terms in the coefficients of  $\widehat{\mathcal{D}}\psi$  for the terms on the left hand side of (37) must be equal to the coefficients of  $\widehat{\mathcal{D}}\psi$  for the terms on the right hand side of (37). We know that  $\omega$  is an ordinary CKY  $p$ -form and satisfies (28), so the sum of the coefficients of  $\widehat{\mathcal{D}}\psi$  in (41) and (42) (corresponding to the left hand side of (37)) can be written as

$$\begin{aligned}
& -(-1)^p \frac{p}{n(p+1)} i_{X_a} d\omega + (-1)^p \frac{p}{n(n-p+1)} e_a \wedge \delta\omega + \frac{p}{2n(p+1)} d\omega \cdot e_a + \frac{p}{2n(n-p+1)} \delta\omega \cdot e_a \\
& + (-1)^p \frac{p}{n^2(p+1)} e_a \cdot e^b \cdot i_{X_b} d\omega - (-1)^p \frac{p}{n^2(n-p+1)} e_a \cdot e^b \cdot (e_b \wedge \delta\omega) \\
& - \frac{p}{2n^2(p+1)} e_a \cdot e^b \cdot d\omega \cdot e_b - \frac{p}{2n^2(n-p+1)} e_a \cdot e^b \cdot \delta\omega \cdot e_b + (-1)^p \frac{p}{2n} (e_a \cdot A + A \cdot e_a) \cdot \omega - (-1)^p \frac{p}{n^2} e_a \cdot A \cdot \omega \\
& = (-1)^p \frac{p}{2n} (e_a \cdot A + A \cdot e_a) \cdot \omega - (-1)^p \frac{p}{n^2} e_a \cdot A \cdot \omega \quad (45)
\end{aligned}$$

where we have used the expansion of the Clifford product in terms of the wedge product and interior derivative as follows

$$\begin{aligned}
d\omega \cdot e_a &= -(-1)^p e_a \wedge d\omega + (-1)^p i_{X_a} d\omega \\
\delta\omega \cdot e_a &= -(-1)^p e_a \wedge \delta\omega + (-1)^p i_{X_a} \delta\omega \quad (46)
\end{aligned}$$

and from the equality  $e^a \cdot \omega \cdot e_a = (-1)^p (n-2p)\omega$

$$\begin{aligned}
e_a \cdot e^b \cdot i_{X_b} d\omega &= (p+1) e_a \wedge d\omega + (p+1) i_{X_a} d\omega \\
e_a \cdot e^b \cdot (e_b \wedge \delta\omega) &= (n-p+1) e_a \wedge \delta\omega + (n-p+1) i_{X_a} \delta\omega \\
e_a \cdot e^b \cdot d\omega \cdot e_b &= -(-1)^p (n-2(p+1)) e_a \wedge d\omega - (-1)^p (n-2(p+1)) i_{X_a} d\omega \\
e_a \cdot e^b \cdot \delta\omega \cdot e_b &= -(-1)^p (n-2(p-1)) e_a \wedge \delta\omega - (-1)^p (n-2(p-1)) i_{X_a} \delta\omega. \quad (47)
\end{aligned}$$

So, the terms that does not contain  $A$  on the left hand side of (45) cancel each other and we obtain the right hand side of (45). On the other hand, one can easily see that, for the sum of the coefficients of  $\widehat{\mathcal{D}}\psi$  in (43) and (44) (corresponding to the right hand side of (37)) is exactly equal to the right hand side of (45). Hence, we prove the first step, that is the coefficients of  $\widehat{\mathcal{D}}\psi$  in the equalities (41)-(44) satisfy (37).

As a second step, we will consider the coefficients of  $\psi$  for the terms in (37) by using (41)-(44). We can write the coefficients of  $\psi$  in (37) in the following form

$$-(-1)^p \frac{p}{2} \omega \cdot K_a + \frac{p}{2(p+1)} \nabla_{X_a} d\omega + \frac{p}{2(n-p+1)} \nabla_{X_a} \delta\omega$$

$$\begin{aligned}
& +(-1)^p \frac{p}{2n} e_a \cdot e^b \cdot \omega \cdot K_b - \frac{p}{2n(p+1)} e_a \cdot e^b \cdot \nabla_{X_b} d\omega - \frac{p}{2n(n-p+1)} e_a \cdot e^b \cdot \nabla_{X_b} \delta\omega \\
& +(-1)^p \frac{p}{(n-1)(n-2)} \omega \cdot e_a \cdot F - (-1)^p \frac{p}{n-2} \omega \cdot i_{X_a} F \\
& -(-1)^p \frac{p}{n(n-1)(n-2)} e_a \cdot e^b \cdot \omega \cdot e_b \cdot F + (-1)^p \frac{p}{n(n-2)} e_a \cdot e^b \cdot \omega \cdot i_{X_b} F \\
& - \frac{p}{4(p+1)} (e_a \cdot A \cdot d\omega + A \cdot e_a \cdot d\omega) - \frac{p}{4(n-p+1)} (e_a \cdot A \cdot \delta\omega + A \cdot e_a \cdot \delta\omega) \\
& + \frac{p}{2n(p+1)} e_a \cdot A \cdot d\omega + \frac{p}{2n(n-p+1)} e_a \cdot A \cdot \delta\omega + \frac{p(n-2)}{4n(p+1)} e_a \cdot A \cdot d\omega \\
& + \frac{p(n-2)}{4n(n-p+1)} e_a \cdot A \cdot \delta\omega + \frac{p}{4(p+1)} A \cdot e_a \cdot d\omega + \frac{p}{4(n-p+1)} A \cdot e_a \cdot \delta\omega \\
& = 0
\end{aligned} \tag{48}$$

and we need to check that the left hand side of (48) is equal to zero. As can easily be seen that the terms that contain  $A$  on the left hand side of (48) cancel each other and we obtain

$$\begin{aligned}
& -(-1)^p p\omega \cdot \left[ \frac{K_a}{2} - \frac{1}{(n-1)(n-2)} e_a \cdot F + \frac{1}{n-2} i_{X_a} F \right] \\
& +(-1)^p \frac{p}{n} e_a \cdot e^b \cdot \omega \cdot \left[ \frac{K_b}{2} - \frac{1}{(n-1)(n-2)} e_b \cdot F + \frac{1}{n-2} i_{X_b} F \right] \\
& + \frac{p}{2(p+1)} \left[ \nabla_{X_a} d\omega - \frac{1}{n} e_a \cdot e^b \cdot \nabla_{X_b} d\omega \right] \\
& + \frac{p}{2(n-p+1)} \left[ \nabla_{X_a} \delta\omega - \frac{1}{n} e_a \cdot e^b \cdot \nabla_{X_b} \delta\omega \right] \\
& = 0.
\end{aligned} \tag{49}$$

We know that  $\omega$  is an ordinary CKY  $p$ -form and it satisfies the integrability conditions in (32)-(34). Considering this and the following equalities (by using (34))

$$\begin{aligned}
e_a \cdot e^b \cdot \nabla_{X_b} d\omega &= -e_a \wedge \delta d\omega - i_{X_a} \delta d\omega \\
&= \frac{(p+1)(n-p)}{p(n-p+1)} e_a \wedge d\delta\omega - i_{X_a} \delta d\omega - \frac{p+1}{p} \left[ e_a \wedge P_b \wedge i_{X^b} \omega + e_a \wedge R_{bc} \wedge i_{X^b} i_{X^c} \omega \right]
\end{aligned} \tag{50}$$

$$\begin{aligned}
e_a \cdot e^b \cdot \nabla_{X_b} \delta\omega &= e_a \wedge d\delta\omega + i_{X_a} d\delta\omega \\
&= e_a \wedge d\delta\omega - \frac{p(n-p+1)}{(p+1)(n-p)} i_{X_a} \delta d\omega + \frac{n-p+1}{n-p} \left[ i_{X_a} (P_b \wedge i_{X^b} \omega) + i_{X_a} (R_{bc} \wedge i_{X^b} i_{X^c} \omega) \right]
\end{aligned} \tag{51}$$

(49) transforms into

$$\begin{aligned}
& -(-1)^p p\omega \cdot \left[ \frac{K_a}{2} - \frac{1}{(n-1)(n-2)} e_a \cdot F + \frac{1}{n-2} i_{X_a} F \right] \\
& +(-1)^p \frac{p}{n} e_a \cdot e^b \cdot \omega \cdot \left[ \frac{K_b}{2} - \frac{1}{(n-1)(n-2)} e_b \cdot F + \frac{1}{n-2} i_{X_b} F \right] \\
& + \frac{1}{2} R_{ba} \wedge i_{X^b} \omega + \frac{1}{2n} \left( e_a \wedge P_b \wedge i_{X^b} \omega + e_a \wedge R_{bc} \wedge i_{X^b} i_{X^c} \omega \right) \\
& + \frac{p}{2(n-p)} \left( (i_{X_b} P_a) i_{X^b} \omega + i_{X_b} R_{ca} \wedge i_{X^c} i_{X^b} \omega \right) \\
& - \frac{p}{2n(n-p)} \left( i_{X_a} (P_b \wedge i_{X^b} \omega) + i_{X_a} (R_{bc} \wedge i_{X^b} i_{X^c} \omega) \right) \\
& = 0.
\end{aligned} \tag{52}$$

In general, this is a very restrictive condition on the curvature characteristics of the manifold and gauge curvature  $F$  related to the ordinary CKY forms of the background. However, a simplification in (52) occurs if we consider the



constant curvature manifolds. In this case, the curvature 2-forms are written as  $R_{ab} = \frac{\mathcal{R}}{n(n-1)}e_a \wedge e_b$  and Ricci 1-forms are  $P_a = \frac{\mathcal{R}}{n}e_a$  while  $K_a = -\frac{\mathcal{R}}{2n(n-1)}e_a$ . By substituting them in (52) and using the equality  $e^a \wedge i_{X_a}\omega = p\omega$ , one can see that the terms that contain curvature characteristics (the terms that does not contain  $F$ ) cancel each other and only the terms that contain  $F$  remain. Moreover, constant curvature manifolds are conformally-flat, namely the conformal 2-forms defined in (18) are equal to zero  $C_{ab} = 0$ . Indeed, in Lorentzian space-times, the gauge curvature  $F$  can be determined from  $C_{ab}$  by using the integrability condition (19) of the gauged twistor equation [4]. Indeed, in constant curvature manifolds, the gauge curvature  $F = 0$  with non-zero  $A$ , so we have flat connections in the definition of the gauged covariant derivative [4, 7]. This means that the remaining terms that contain  $F$  in (52) will be equal to zero and the vanishing condition of the left hand side of (52) is satisfied. Hence, the terms that are in the coefficients of  $\psi$  satisfy (37).

In that way, we prove that the operator defined in (35) is a symmetry operator for the gauged twistor equation in constant curvature manifolds. So, one can construct gauged twistor spinors which are the supersymmetry generators of supersymmetric field theories coupled to supergravity from a known solution by using the ordinary CKY forms of the constant curvature background through (35). The constant curvature manifolds such as anti-de Sitter (AdS) space-times are important since they occur in the backgrounds of supergravity theories and supersymmetric field theories.

### A. From ordinary twistors to gauged twistors

The construction of the symmetry operator (35) gives rise to a relation between ordinary twistor spinors and gauged twistor spinors. The spinor bilinears of ordinary twistor spinors correspond to the ordinary CKY forms [23]. This means that the symmetry operators in (35) can be written in terms of the ordinary twistor spinors. For an ordinary twistor spinor  $\phi$ , the  $p$ -form Dirac currents as defined in (21)

$$(\phi\bar{\phi})_p = (\phi, e_{a_p \dots a_{2p+1}} \cdot \phi) e^{a_1 a_2 \dots a_p} \quad (53)$$

can be replaced with the CKY  $p$ -forms in the symmetry operator (35). So, the symmetry operator of a gauged twistor spinor  $\psi$  can be constructed from the  $p$ -form Dirac currents of ordinary twistor spinors as

$$L_{\phi\bar{\phi}}\psi = -(-1)^p \frac{p}{n} (\phi\bar{\phi})_p \cdot \hat{\mathcal{D}}\psi + \frac{p}{2(p+1)} d(\phi\bar{\phi})_p \cdot \psi + \frac{p}{2(n-p+1)} \delta(\phi\bar{\phi})_p \cdot \psi \quad (54)$$

and this means that the ordinary twistor spinors generate the solutions of the gauged twistor equation. The exterior derivative and co-derivative of the  $p$ -form Dirac currents of the ordinary twistor spinors can be found as [23]

$$d(\phi\bar{\phi})_p = \frac{p+1}{n} \left( \not{d}(\phi\bar{\phi}) - 2i_{X^a}(\phi\bar{\nabla}_{X_a}\phi) \right)_{p+1} \quad (55)$$

$$\delta(\phi\bar{\phi})_p = -\frac{n-p+1}{n} \left( \not{d}(\phi\bar{\phi}) - 2e^a \wedge (\phi\bar{\nabla}_{X_a}\phi) \right)_{p-1} \quad (56)$$

and (54) can also be written in the following form

$$\begin{aligned} L_{\phi\bar{\phi}}\psi &= -\frac{p}{n} \left[ (-1)^p (\phi\bar{\phi})_p \cdot \hat{\mathcal{D}}\psi + \frac{1}{n} \left( (i_{X^a}(\phi\bar{e}_a \cdot \not{D}\phi))_{p+1} - (e^a \wedge (\phi\bar{e}_a \cdot \not{D}\phi))_{p-1} \right) \cdot \psi \right] \\ &\quad + \frac{p}{2n} \left[ (\not{d}(\phi\bar{\phi}))_{p+1} - (\not{d}(\phi\bar{\phi}))_{p-1} \right] \cdot \psi \end{aligned} \quad (57)$$

For  $A = 0$ , this reduces to the symmetry operators of ordinary twistor spinors written in terms of ordinary twistor spinors [17].

The symmetry operators of the gauged twistor equation is defined in constant curvature manifolds and the set of CKY forms in that case is of maximal dimension. The maximum number of CKY  $p$ -forms in  $n$  dimensions is [24]

$$C_p = \binom{n}{p-1} + 2 \binom{n}{p} + \binom{n}{p+1} \quad (58)$$

and the dimension of the space of ordinary twistor spinors is given in  $n$  dimensional constant curvature manifolds as [26]

$$t = 2^{\lfloor n/2 \rfloor} + 1. \quad (59)$$

Those sets of CKY forms or ordinary twistor spinors generate the gauged twistor spinors through (35) or (54).

## V. CONCLUSION

Symmetry operators of the gauged twistor equation in terms of ordinary CKY forms is constructed in constant curvature backgrounds. Since the existence of CKY forms or gauged twistor spinors is a restrictive condition on the underlying manifold, the construction of symmetry operators is constrained to constant curvature manifolds. This is expected, because of the fact that the constant curvature manifolds have maximum numbers of CKY forms and ordinary twistor spinors and constructing symmetry operators using them in constant curvature manifolds is more possible than the other cases. Construction of those symmetry operators provides a new way to obtain the supersymmetry generators of supersymmetric and superconformal field theories in curved backgrounds.

The spinor bilinears of gauged twistor spinors correspond to gauged CKY forms. However, the symmetry operators contain ordinary CKY forms and not gauged CKY forms. This means that the extended superalgebras that contain gauged twistor spinors and gauged CKY forms cannot be obtained by using the constructed symmetry operators while they can be constructed for the cases of ordinary twistor spinors and geometric Killing spinors [16, 17]. So, one can search for the other types of symmetry operators of gauged twistor equation constructed out of gauged CKY forms and try to construct extended superalgebras from them. These superalgebra structures are important in the classification problem of supergravity and supersymmetric field theory backgrounds.

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